On the birational geometry of toric Fano 4-folds

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Abstract . In this Note, we announce a factorization result for equivariant birational morphisms between toric 4-folds whose source is Fano: such a morphism is always a composite of blow-ups along smooth invariant centers. Moreover, we show with a counterexample that, differently from the 3-dimensional case, even if both source and target are Fano, the intermediate varieties can not be chosen Fano.

Sur la géométrie birationnelle des variétés toriques de Fano de dimension 4

Résumé . *Dans cette Note, nous annonçons un résultat de factorisation pour les morphismes birationnels équivariants entre variétés toriques de dimension 4, ayant comme source une variété de Fano: un tel morphisme est toujours donné par une suite d'éclatements le long de sous-variétés lisses invariantes. Nous montrons à l'aide d'un contrexemple que, à la différence du cas de la dimension 3, même si la source et le but sont de Fano, les variétés intermédiaires ne peuvent pas être choisies de Fano.*

Version française abrégée

Dans cette Note, on s'intéresse aux morphismes birationnels équivariants entre variétés toriques de dimension 4, dans le cas où la source est une variété de Fano. En dimension 3 Sato [9], grâce à l'étude des applications birationnelles entre variétés toriques de Fano, obtient d'une nouvelle façon la classification des variétés toriques de Fano de dimension 3 et complète la classification en dimension 4. En particulier il obtient le:

Théorème 2.1 (Sato [9]). — *Soient Y et X deux variétés toriques de Fano de dimension 3 et $f\colon Y \to X$ un morphisme birationnel équivariant. Alors il existe une suite*

$$Y = X_r \xrightarrow{\varphi_r} X_{r-1} \xrightarrow{\varphi_{r-1}} \cdots\cdots \xrightarrow{\varphi_2} X_1 \xrightarrow{\varphi_1} X_0 = X$$

telle que pour tout $i = 1, \dots, r$ φ_i est un éclatement le long d'une sous-variété invariante lisse, X_i est une variété torique de Fano et $f = \varphi_1 \circ \ldots \circ \varphi_r$.

Rappelons que en général un morphisme birationnel ne se factorise pas comme une suite d'éclatements le long de sous-variétés lisses, même dans le cas torique. La démonstration du théorème 2.1 est obtenue en deux étapes:
1) en supposant Y de Fano, on montre que $f\colon Y \to X$ admet une factorisation en une suite d'éclatements le long de sous-variétés lisses invariantes;
2) on montre que si de plus X est de Fano, toutes les variétés intermédiaires doivent être de Fano.

En dimension 4 nous montrons que seul 1) reste vrai:

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Théorème 2.2. — *Soient Y et X deux variétés toriques lisses, complètes, de dimension 4, et $f\colon Y \to X$ un morphisme birationnel équivariant. Si Y est de Fano, alors il existe une suite*

$$Y = X_r \xrightarrow{\varphi_r} X_{r-1} \xrightarrow{\varphi_{r-1}} \cdots\cdots \xrightarrow{\varphi_2} X_1 \xrightarrow{\varphi_1} X_0 = X \qquad (\heartsuit)$$

où pour tout $i = 1, \ldots, r$ φ_i est un éclatement le long d'une sous-variété lisse invariante, X_i est une variété torique complète lisse et $f = \varphi_1 \circ \ldots \circ \varphi_r$.

On montre ce résultat à l'aide du procédé utilisé par Sato en dimension 3: on fixe un cône σ de dimension 4 dans Σ_X et on étudie les subdivisions possibles de σ dans Σ_Y. Le fait que Y soit de Fano impose beaucoup de restrictions sur la combinatoire de l'éventail:

Proposition 2.3. — *Soient Y et X deux variétés toriques lisses, complètes, de dimension 4, et $f\colon Y \to X$ un morphisme birationnel équivariant. On suppose Y de Fano et on fixe un cône $\sigma \in \Sigma_X$ de dimension 4. Alors il y a seulement 17 subdivisions possibles de σ dans Σ_Y. De ces 17 subdivisions possibles, 4 apparaissent uniquement si $X \simeq \mathbf{P}^4$.*

Les démonstrations du théorème 2.2 et de la proposition 2.3 apparaîtront ailleurs.

Le point 2) est en défaut dès la dimension 4:

Proposition 3.1. — *Il existe deux variétés toriques de Fano de dimension 4 Y et X, et un morphisme birationnel équivariant $f\colon Y \to X$, tels que f ne se factorise pas comme une suite d'éclatements comme dans $(\heartsuit)$, avec X_i de Fano pour tout $i = 1, \ldots, r$.*

Le morphisme f est la composition de deux éclatements $Y \to W \to X$, où X est $\mathbf{P}^4$ éclaté le long d'une droite.

Toric Fano varieties are classified in dimension less or equal than 4: $\mathbf{P}^1$ is the only Fano curve, and there are 5 toric Fano surfaces, 18 toric Fano 3-folds [1,10] and 124 toric Fano 4-folds [3,9]. In [9], Sato shows some interesting factorization properties of equivariant birational maps and morphisms between low-dimensional toric Fano varieties. With these results, he obtains a new proof of the classification in dimension 3 and completes the classification in dimension 4. In particular, he shows that every birational equivariant morphism between toric Fano 3-folds is a composite of blow-ups with smooth invariant centers between toric Fano 3-folds (theorem 2.1), and addresses the question whether similar results hold in higher dimension. We show, with a counterexample, that the same result is false in dimension 4, and announce a weaker factorization result in dimension 4 (theorem 2.2).

In the first section we recall some notions of toric geometry, in particular the definitions of primitive collection and primitive relation, and their link with toric Mori theory. The second section concerns factorization of birational equivariant morphisms between toric Fano varieties, and the third one contains our counterexample.

1. Preliminaries in toric geometry.

For all the standard results in toric geometry, we refer to [5] and [7].

Let X be an n-dimensional toric variety: X is described by a finite fan Σ_X in the vector space $N_{\mathbf{Q}} = N \otimes_{\mathbf{Z}} \mathbf{Q}$, where N is a free abelian group of rank n. We'll always assume X smooth and complete, so the support of Σ_X is the whole space $N_{\mathbf{Q}}$ and every cone in Σ_X is generated by

a part of a basis of N. We remember that for each $r = 0, \dots, n$ there is a bijection between the cones of dimension r in Σ_X and the orbits of codimension r in X; we'll denote by $V(\sigma)$ the closure of the orbit corresponding to $\sigma \in \Sigma_X$, and $V(x) = V(\langle x \rangle)$ in case of 1-dimensional cones $\langle x \rangle \in \Sigma_X$. For each 1-dimensional cone $\rho \in \Sigma_X$, let $v_\rho \in \rho \cap N$ be its primitive generator, and $G(\Sigma_X) = \{v_\rho \mid \rho \in \Sigma_X, \dim \rho = 1\}$ the set of all generators in Σ_X.

Primitive collections. — The language of primitive collections and primitive relations was introduced by Batyrev ([2, 3]); it is particularly convenient to describe fans of toric Fano varieties.

Definition 1.1. — A subset $P = \{x_1, \dots, x_h\} \subseteq G(\Sigma_X)$ is a *primitive collection* for Σ_X if $\langle x_1, \dots, x_h \rangle \notin \Sigma_X$, but $\langle x_1, \dots, \check{x}_i, \dots, x_h \rangle \in \Sigma_X$ for each $i = 1, \dots, h$.

We denote by $\mathrm{PC}(\Sigma_X)$ the set of all primitive collections for Σ_X.

Definition 1.2. — Let $P = \{x_1, \dots, x_h\} \subseteq G(\Sigma_X)$ be a primitive collection. Since X is complete, the point $x_1 + \cdots + x_h$ is contained in some cone of Σ_X; let $\sigma_P = \langle y_1, \dots, y_k \rangle$ be the unique cone in Σ_X such that $x_1 + \cdots + x_h \in \operatorname{Rel\,Int} \sigma_P$, the relative interior of σ_P. Then we get a linear relation

$$x_1 + \cdots + x_h - (a_1 y_1 + \cdots + a_k y_k) = 0$$

with a_i a positive integer for each $i = 1, \dots, k$. We call this relation the *primitive relation* associated to P. The *degree* of P is the integer $\deg P = h - a_1 - \cdots - a_k$.

Let $\mathcal{A}_1(X)$ be the group of algebraic 1-cycles on X modulo numerical equivalence, $\mathcal{N}_1(X) = \mathcal{A}_1(X) \otimes_{\mathbf{Z}} \mathbf{Q}$, and $\mathrm{NE}(X) \subset \mathcal{N}_1(X)$ the cone of Mori, generated by classes of effective curves. The $\mathbf{Q}$-vector space $\mathcal{N}_1(X)$ has dimension ρ_X, the Picard number of X. We recall the well-known result:

Proposition 1.3. — *The group $\mathcal{A}_1(X)$ is canonically isomorphic to the lattice of integral relations among the elements of $G(\Sigma_X)$.*

A relation $\sum_{x \in G(\Sigma_X)} a_x x = 0$ corresponds to a class in $\mathcal{A}_1(X)$ whose intersection with $V(x)$ is a_x, for all $x \in G(\Sigma_X)$. By proposition 1.3, for every primitive collection $P \in \mathrm{PC}(\Sigma_X)$, the associated primitive relation defines a class $r(P) \in \mathcal{A}_1(X)$. Since the canonical class on X is given by $K_X = -\sum_{x \in G(\Sigma_X)} V(x)$, for every primitive collection P we have $-K_X \cdot r(P) = \deg P$.

Some links with toric Mori theory. — The cone of effective curves in a complete toric variety has been studied by Reid in [8]: it is a closed, polyhedral cone, generated by classes of invariant curves. Moreover, it follows from Kleiman's criterion of ampleness [6] that X is projective if and only if the cone $\mathrm{NE}(X)$ is strictly convex; in this case, its extremal rays are spanned by invariant curves.

Theorem 1.4 (Reid [8], Batyrev [2]). — *The cone of effective curves $\mathrm{NE}(X)$ is generated by primitive relations:*

$$\mathrm{NE}(X) = \sum_{P \in \mathrm{PC}(\Sigma_X)} \mathbf{Q}_{\geq 0}\, r(P).$$

This gives an important characterization of toric Fano varieties:

Proposition 1.5 (Batyrev [3]). — *The variety X is Fano if and only if all primitive collections in Σ_X have strictly positive degree.*

Blow-ups. — By a smooth equivariant blow-up we mean the blow-up of a smooth toric variety along an invariant, smooth subvariety. The resulting variety is clearly a smooth toric variety.

Let $f\colon Y \to X$ be a smooth equivariant blow-up, along a subvariety $V(\tau) \subset X$, $\tau = \langle x_1, \ldots, x_h \rangle$. We recall that Σ_Y is a subdivision of Σ_X, therefore $G(\Sigma_X) \subseteq G(\Sigma_Y)$. The set $P = \{x_1, \ldots, x_h\}$ is a primitive collection for Σ_Y, with relation

$$r(P)\colon \quad x_1 + \cdots + x_h = x. \tag{$\star$}$$

The divisor $V(x)$ is the exceptional divisor in Y and $r(P)$ corresponds to the numerical class of a $\mathbf{P}^1$ contained in a fiber of f. In general, $r(P)$ needs not to be extremal in $\mathrm{NE}(Y)$:

Proposition 1.6 (Bonavero [4]). — *Suppose Y that is projective: then $r(P)$ generates an extremal ray in $\mathrm{NE}(Y)$ if and only if X is projective.*

2. Results on birational morphism between toric Fano varieties.

In [9], Sato uses primitive collections and primitive relations to study equivariant birational maps between toric Fano varieties. In this way, he shows a strong property of birational morphisms between toric Fano 3-folds, without using the classification:

Theorem 2.1 (Sato [9]). — *Let Y and X be two toric Fano 3-folds, and $f\colon Y \to X$ a birational equivariant morphism. Then there exists a sequence of smooth equivariant blow-ups*

$$Y = X_r \xrightarrow{\varphi_r} X_{r-1} \xrightarrow{\varphi_{r-1}} \cdots\cdots \xrightarrow{\varphi_2} X_1 \xrightarrow{\varphi_1} X_0 = X$$

such that for each $i = 1, \ldots, r$ X_i is a toric Fano 3-fold and $f = \varphi_1 \circ \ldots \circ \varphi_r$.

We remark that in general it is not possible to factorize a birational morphism in a sequence of smooth blow-ups; here not only it is possible, but also all the intermediate varieties are Fano. In fact, theorem 2.1 is obtained in two steps:
1) given a birational equivariant morphism $f\colon Y \to X$, if Y is Fano, it is always possible to find a factorization of f in a sequence of smooth, equivariant blow-ups;
2) if moreover X is Fano, all the intermediate varieties must be Fano.

In the 4-dimensional case, only the first part of this result stands true:

Theorem 2.2. — *Let Y and X be nonsingular, complete toric varieties of dimension 4, and $f\colon Y \to X$ a birational equivariant morphism. Suppose Y is Fano: then there exists a sequence of smooth equivariant blow-ups*

$$Y = X_r \xrightarrow{\varphi_r} X_{r-1} \xrightarrow{\varphi_{r-1}} \cdots\cdots \xrightarrow{\varphi_2} X_1 \xrightarrow{\varphi_1} X_0 = X \tag{$\heartsuit$}$$

such that for each $i = 1, \ldots, r$ φ_i is a smooth equivariant blow-up, X_i is a nonsingular, complete toric variety and $f = \varphi_1 \circ \ldots \circ \varphi_r$.

The proof of this result uses the same idea of Sato's proof in the 3-dimensional case: for a fixed cone $\sigma \in \Sigma_X$, we study the possible subdivisions of σ in Σ_Y. The hypothesis that Y is Fano gives many combinatorial restrictions on the possible subdivisions:

Proposition 2.3. — *Let Y be a toric Fano 4-fold, X a nonsingular, complete toric 4-fold and $f\colon Y \to X$ a birational equivariant morphism. We fix a 4-dimensional cone $\sigma \in \Sigma_X$. Then there are only 17 possible subdivisions of σ in Σ_Y, 4 of which can appear only if $X \simeq \mathbf{P}^4$.*

The proofs of theorem 2.2 and proposition 2.3 will appear elsewhere.

3. The example.

In this section we are going to show the

Proposition 3.1. — *There exist two nonsingular toric Fano 4-folds Y and X and a birational equivariant morphism $f\colon Y \to X$, such that f doesn't admit a decomposition in smooth equivariant blow-up as in ($\heartsuit$) with X_i a toric Fano 4-fold for all $i = 1, \dots, r$.*

Proof. — Fix two lines L_1, L_2 in $\mathbf{P}^4$ which intersect in a point. If $\{e_1, e_2, e_3, e_4\}$ is a basis of N, then we have $G(\Sigma_{\mathbf{P}^4}) = \{e_0, e_1, e_2, e_3, e_4\}$ with the only primitive relation $e_0 + e_1 + e_2 + e_3 + e_4 = 0$. We can suppose that $L_1 = V(\langle e_1, e_2, e_3 \rangle)$, $L_2 = V(\langle e_2, e_3, e_4 \rangle)$; let X be the blow-up of $\mathbf{P}^4$ along L_1.

In [9], Sato studies how primitive collections vary under smooth equivariant blow-ups and blow-downs. Let $f\colon Z \to W$ be a blow-up: Sato gives an algorythm to compute explicitly the primitive collections of Σ_Z from the ones of Σ_W, and viceversa. We have used this algorythm to obtain the primitive collections of X and of the varieties we'll introduce in the sequel.

The fan Σ_X of X in $N_{\mathbf{Q}}$ has vertices $\mathrm{G}(\Sigma_X) = \{e_0, \dots, e_5\}$ and its primitive relations are $e_1 + e_2 + e_3 = e_5$ and $e_0 + e_4 + e_5 = 0$. These relations are both extremal; $\mathcal{N}_1(X)$ is a vector space of dimension $\rho_X = 2$ and $\mathrm{NE}(X) \subset \mathcal{N}_1(X)$ is a 2-dimensional cone. Furthermore, the primitive relations have respectively degree 2 and 3, so X is a nonsingular toric Fano 4-fold, by proposition 1.5. In X we have the invariant subvarieties:

$\widetilde{L_2} = V(\langle e_2, e_3, e_4 \rangle)$, the strict transform of L_2;
$D = V(e_5) \simeq \mathbf{P}^2 \times \mathbf{P}^1$, the exceptional divisor;
$S = V(\langle e_4, e_5 \rangle)$, the $\mathbf{P}^2$ contained in D passing through the point $p = \widetilde{L_2} \cap D$;
$C_1 = V(\langle e_2, e_4, e_5 \rangle)$ and $C_2 = V(\langle e_3, e_4, e_5 \rangle)$, the two invariant lines in S passing through p.

Now we blow up X along the curve $\widetilde{L_2}$. We get a toric 4-fold W whose fan Σ_W has vertices $\mathrm{G}(\Sigma_W) = \{e_0, \dots, e_6\}$ and primitive relations:

$$\begin{array}{lllllll} (E) & e_2 + e_3 + e_4 & = & e_6 & e_1 + e_2 + e_3 & = & e_5 \\ (E) & e_1 + e_6 & = & e_4 + e_5 & e_0 + e_4 + e_5 & = & 0 \\ (E) & e_0 + e_5 + e_6 & = & e_2 + e_3 & & & \end{array}$$

Here the letter E stands for extremal; the decomposition of the last two relations are:

$$r(\{e_1, e_2, e_3\}) = r(\{e_1, e_6\}) + r(\{e_2, e_3, e_4\}), \quad r(\{e_0, e_4, e_5\}) = r(\{e_0, e_5, e_6\}) + r(\{e_2, e_3, e_4\}).$$

Therefore $\mathrm{NE}(W)$ is a simplicial cone in $\mathcal{N}_1(W)$, which has dimension $\rho_W = 3$. The variety W is not Fano: indeed, there is a primitive collection $\{e_1, e_6\}$ of degree zero. The invariant curves whose numerical class in $\mathrm{NE}(W)$ corresponds to the relation $e_1 + e_6 = e_4 + e_5$ are exactly $\widetilde{C_1} = V(\langle e_2, e_4, e_5 \rangle)$ and $\widetilde{C_2} = V(\langle e_3, e_4, e_5 \rangle)$, the strict transforms of C_1 and C_2, both contained in the surface $\widetilde{S} = V(\langle e_4, e_5 \rangle)$, which is now a $\mathbf{P}^2$ blown-up in one point, *i.e.* the Hirzebruch surface $\mathbf{F}_1$. The curves $\widetilde{C_1}$ and $\widetilde{C_2}$ have anticanonical degree zero.

Finally, let Y be the blow-up of W along the surface $\widetilde{S}$. The fan Σ_Y of Y in N has vertices $\mathrm{G}(\Sigma_Y) = \{e_0, \dots, e_7\}$ and primitive relations:

$$\begin{array}{llllllllllll} (E) & e_4 + e_5 & = & e_7 & (E) & e_2 + e_3 + e_7 & = & e_5 + e_6 & e_1 + e_2 + e_3 & = & e_5 \\ (E) & e_1 + e_6 & = & e_7 & (E) & e_0 + e_7 & = & 0 & e_2 + e_3 + e_4 & = & e_6. \end{array}$$

There are 4 extremal classes, and $\mathcal{N}_1(Y)$ has dimension $\rho_Y = 4$, hence the cone $\mathrm{NE}(Y) \subset \mathcal{N}_1(Y)$ is again simplicial. The decompositions of the last two relations are:

$$r(\{e_1, e_2, e_3\}) = r(\{e_1, e_6\}) + r(\{e_2, e_3, e_7\}), \quad r(\{e_2, e_3, e_4\}) = r(\{e_4, e_5\}) + r(\{e_2, e_3, e_7\}).$$

Since all primitive relations have strictly positive degree, proposition 1.5 implies that Y is a Fano variety. *We claim that Y doesn't admit any equivariant blow-down to a nonsingular toric Fano 4-fold.*

Indeed, any equivariant blow-down of Y to a smooth toric 4-fold gives a primitive relation as in $(\star)$; in Σ_Y there are 4 primitive relations of this type.

The two relations $e_4 + e_5 = e_7$ and $e_1 + e_6 = e_7$ give blow-downs respectively to W and $\overline{W}$, where $\overline{W}$ is obtained from $\mathbf{P}^4$ blowing-up first L_2 and then the strict transform $\widetilde{L_1}$; $\overline{W}$ is clearly isomorphic to W, therefore it is not Fano.

The other two relations to consider are $e_2 + e_3 + e_4 = e_6$ and $e_1 + e_2 + e_3 = e_5$. Suppose, for instance, that $e_1 + e_2 + e_3 = e_5$ comes from an equivariant blow-up, as in $(\star)$. Then every cone of Σ_Y containing e_5 must contain also two generators among $\{e_1, e_2, e_3\}$. On the other hand, by Reid's study of the geometry of a fan around an extremal wall in [8], the fact that $e_2 + e_3 + e_7 = e_5 + e_6$ is extremal implies $\langle e_3, e_5, e_6, e_7 \rangle \in \Sigma_Y$; this gives a contradiction. The same for $e_2 + e_3 + e_4 = e_6$.

Therefore, the birational morphism $Y \to X$ given by the composition of the two blow-ups $Y \to W \to X$ is the example we were looking for. ∎

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